SPLICE DIAGRAM DETERMINING SINGULARITY LINKS AND UNIVERSAL ABELIAN COVERS

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Abstract. To a rational homology sphere graph manifold one can associate a weighted tree invariant called splice diagram. In this article we prove a sufficient numerical condition on the splice diagram for a graph manifold to be a singularity link. We also show that if two manifolds have the same splice diagram, then their universal abelian covers are homeomorphic. To prove the last theorem we have to generalize our notions to orbifolds.

1. Introduction

For prime 3-manifolds M one has several decomposition theorems, like the geometric decomposition which cuts M along embedded tori and Klein bottles into geometric pieces, or the JSJ decomposition which cut M along embedded tori into simple and Seifert fibered pieces. A graph manifold is a manifold that does not have any hyperbolic pieces in its geometric decomposition, or equivalently only has Seifert fibered pieces in its JSJ decomposition. To a graph manifold one can associate several graph invariants, and in this paper we are going to show properties of one of these called the splice diagram.

Splice diagrams were original introduced in [EN85] and [Sie80] only for manifolds that are integer homology spheres. Splice diagrams were then generalized to rational homology spheres in [NW02] and used extensively in [NW05b] and [NW05a]. Our splice diagrams differs from the ones in [EN85] in that we do not allow negative weights on edges, and from the ones in [NW05b], [NW05a] and [NW02] in that we have decorations on the nodes, but it is shown in [NW05a] that in the case of singularity links their splice diagrams are the same as ours.

Now it has long been known that the link of a isolated complex surface singularity is a graph manifold who has a plumbing diagram with only orientable base surfaces and negative definite intersection form. Grauert showed that the inverse is also true. It is shown in appendix 2 of [NW05a] that a rational homology sphere singularity link has a splice diagram without any negative decorations at nodes and has all edge determinants positive. We here prove the other direction to get the following theorem.

Theorem 1. *Let M be a rational homology sphere graph manifold with splice diagram Γ. Then M is a singularity link if and only if Γ has no negative decorations at nodes and all edge determinants are positive.*

Another interesting thing in the study of 3 manifolds is knowledge of the abelian covers of the manifold. Since our manifolds are rational homology spheres their universal abelian covers are finite covers, and we show that the splice diagram of a manifold determines its universal abelian cover, more precisely we prove.

Theorem 2. *Let M and M' be rational homology sphere graph manifolds with the same splice diagram Γ. Then M and M' have isomorphic universal abelian covers.*

1991 *Mathematics Subject Classification.* 32S28, 32S50, 57M10, 57M27.

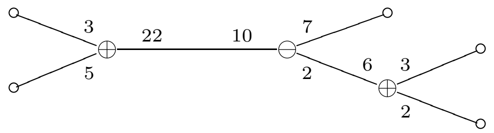
Key words and phrases. surface singularity, rational homology sphere, abelian cover, orbifolds.

In 2 we define our splice diagrams and prove several facts we need about them, among other how one gets a splice diagram from a plumbing diagram of the manifold. In 3 we show some important lemmas for our proofs, and that the splice diagram together with the order of the first homology group determines the decomposition graph of [Neu97]. In 4 we then prove the first theorem above. In 5 we define graph orbifolds and show some of their properties, since we need them in the proof of the second theorem which we prove in 6.

The author wish to thank Walter Neumann for advice with the article.

2. Preliminaries on Splice Diagrams

A splice diagram is a tree which has vertices of valence one, which we call leaves, and vertices of valence greater than or equal to 3, which are called nodes. The end of an edge adjacent to a node of the splice diagram is decorated with a non negative integer, and each node is decorated with either a plus or a minus sign; in general one only writes the minus signs. Here is an example:



Given a rational homology sphere graph manifold M, one makes the splice diagram $\Gamma(M)$ by taking a node for each Seifert fibered piece. Then one attaches an edge between two nodes if they are glued along a torus, and one adds leaves (vertex attached to a node along a edge) to each node, one for each singular fiber of the Seifert fibered piece corresponding to the node.

The decoration d_{ve} on edge e at node v one gets by cutting the manifold along the torus corresponding to e, gluing a solid torus in the piece not containing the Seifert fibered piece corresponding to v, by identifying a meridian of the solid torus with the fiber of in the Seifert fibered piece and a longitude with a simple closed curve, which is given by the JSJ decomposition. Then the d_{ve} is the order of the first homology of this new manifold. We assign 0 if the homology is infinite.

For the decorations on nodes, we need the following definition.

Definition 2.1. Let $L_0, L_1 \subset M$ be two knots in a rational homology sphere. Let $C_1 \subset M$ be a submanifold, such that $\partial C_1 = d_1 L_1$, for some integer d_1. Then the linking number of L_0 with L_1 is defined to be $\mathrm{lk}(L_0, L_1) = \frac{1}{d_0} L_0 \bullet C_1$. Where $\bullet$ denotes the intersection product in M.

To see that $\mathrm{lk}(L_0, L_1)$ is well defined, we need to show that if $C_1' \subset M$ is a submanifold such that $\partial C_1' = d_1' L_1$ then $\frac{1}{d_0} L_0 \bullet C_1 = \frac{1}{d_0'} L_0 \bullet C_1'$, a C_1 always exist since M is a rational homology sphere. Since $\partial C_1' = d_1' L_1$ we have that $\partial(d_1 C_1') = d_1 d_1' L_1$, in the same way we have that $\partial(d_1' C_1) = d_1 d_1' L_1$. We can thus form a closed submanifold $N = d_1 C_1' \bigcup_{d_1 d_1' L_0} -d_1' C_1$. Since M is a rational homology sphere then the homology class of N is 0, so $L_0 \bullet N = 0$. But then $0 = L_0 \bullet N = L_0 \bullet (d_1 C_1' \bigcup_{d_1 d_1' L_0} -d_1' C_1) = L_0 \bullet d_1 C_1' - L_0 \bullet d_1' C_1$. Since the intersection product is bilinear we get the result by dividing by $d_1 d_1'$.

To show that $\mathrm{lk}(L_0, L_1) = \mathrm{lk}(L_1, L_0)$, we will define another notion of linking number to show that this is symmetric and equal to our first definition.

Definition 2.2. Let $L_0, L_1 \subset M$ be knots in a rational homology sphere, let X be a compact 4-manifold such that $M = \partial X$. Let $A_0, A_1 \subset X$ be submanifolds such that $\partial A_i = d_i L_i$ for some integers d_0, d_1, and such that A_1 has zero intersection

with any 2-cycle in X. Then let $\widetilde{\mathrm{lk}}(L_0, L_1) = \frac{1}{d_0 d_1} A_0 \cdot A_1$, where $\cdot$ denotes the intersection product in X.

A_i exists since M is a rational homology sphere, and we can the choose $A_i \subset M$. That A_0 can be chosen so that it does not intersect any closed cycles of X follows because a collar neighbourhood of M has zero second homology, since $H_2((0,1] \times M) \cong H_2(M) = \{0\}$, and hence we can just choose $A_0 \subset (0,1] \times M$. To show $\widetilde{\mathrm{lk}}(L_0, L_1)$ is well defined we start by showing that if $A_1' \subset X$ is such that $\partial A_1' = d_1' L_1$ then $\frac{1}{d_0 d_1} A_0 \cdot A_1 = \frac{1}{d_0 d_1'} A_0 \cdot A_1'$. We form $N = (d_1' A_1 \bigcup_{d_1 d_1' L_1} -d_1 A_1')$, then $A_0 \cdot N = 0$ since N is a closed 2-cycle, and it follows that $\frac{1}{d_0 d_1} A_0 \cdot A_1 = \frac{1}{d_0 d_1'} A_0 \cdot A_1'$ as above. Now assume $A_0' \subset X$ is such that $\partial A_0' = d_0' L_0$ and A_0' has zero intersection with all 2-cycles in X. To show that $\frac{1}{d_0 d_1} A_0 \cdot A_1 = \frac{1}{d_0' d_1} A_0' \cdot A_1$ we can choose a $A_1 \subset X$ such that A_1 has zero intersection with any 2-cycles, since changing A_1 does not change $\widetilde{\mathrm{lk}}(L_0, L_1)$ as just shown. Then form $N' = (d_0' A_0 \bigcup_{d_0 d_0' L_0} -d_0 A_0')$. Now N is a 2-cycle and by our choice of A_1, $N \cdot A_1 = 0$ it follows that $\frac{1}{d_0 d_1} A_0 \cdot A_1 = \frac{1}{d_0' d_1} A_0' \cdot A_1$ and therefor that $\widetilde{\mathrm{lk}}(L_0, L_1)$ is well defined.

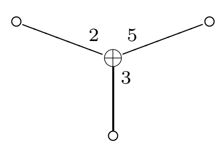
That $\widetilde{\mathrm{lk}}(L_0, L_1)$ is symmetric is clear since the intersection product of 2 dimensional submanifolds of a 4 dimensional manifold is symmetric, and that it is the first of the two that has zero intersection with the 2-cycles does not matter since we could have chosen both A_0 and A_1 to have zero intersection with any 2-cycles, making the definition symmetric.

Proposition 2.3. $\mathrm{lk}(L_0, L_1) = \widetilde{\mathrm{lk}}(L_0, L_1)$

Proof. We choose A_0 such that in the collar neighborhood $(0,1] \times M$ we have that $A_0 \cap (0,1] \times M = (0,1] \times d_0 L_0$, and we choose A_1 such that $A_1 \subset M$. Then we get that $\widetilde{\mathrm{lk}}(L_0, L_1) = \frac{1}{d_0 d_1} (0,1] \times d_0 L_0 \cdot A_1 = \frac{1}{d_0 d_1} \{1\} \times d_0 L_0 \cdot A_1 = \frac{d_0}{d_0 d_1} L_0 \bullet A_1 = \mathrm{lk}(L_0, L_1)$ ∎

Hence $\mathrm{lk}(L_0, L_1)$ is symmetric.

To finish the decoration of the splice diagram we add at a node v a sign ε_v corresponding to the sign of the linking number of two non singular fibers in the Seifert fibration. So by this we can get that the splice diagram for the Poincare homology sphere is the following



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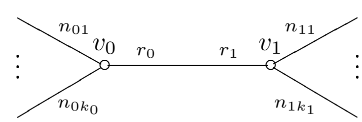
An edge between two nodes look like



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and to such edges we assign a number called the edge determinant.

Definition 2.4. The edge determinant D associated to a edge between nodes v_0 and v_1 is defined by the following equation.

$$D = r_0 r_1 - \varepsilon_0 \varepsilon_1 N_0 N_1. \tag{1}$$

where N_i is the product of all the edge weights adjacent to v_i, except the one on the edge between v_0 and v_1, r_i is the edge weight adjacent to v_i on the edge between

v_0 and v_1, and ε_i is the sign on the node v_i, if we interpret a plus sign as 1 and a minus sign as -1.

One way of getting a splice diagram for a manifold is from a plumbing diagram for the manifold. Suppose M has a plumbing diagram $\Delta(M)$ which we assume is in normal form, which means that all base surfaces are orientable and the decorations on strings are less that or equal to -2. This is not quite the same normal form as in [Neu81] but using the plumbing calculus of that article we can get from Neumann's normal form to the one we use see [Neu89b].

We then get a splice diagram $\Gamma(M)$ by taking one node for each node in $\Delta(D)$, i.e. a vertex with more than 3 edges or genus $\neq 0$. Since we are only working with rational homology spheres, every vertex of the plumbing diagram has genus $= 0$. Connect two nodes in $\Gamma(D)$ if there is a string between the corresponding nodes in $\Delta(M)$, and add a leaf at a node in $\Gamma(M)$ for each string starting at that node in $\Delta(M)$ and not ending at any node.

If $\Delta(M)$ is a plumbing of a manifold we denote the intersection matrix by $A(\Delta(M))$ and let $\det(\Delta(M)) = \det(-A(\Delta(M)))$.

Lemma 2.5. *Let v be a node in $\Gamma(M)$, and e be a edge on that node. We get the weight d_{ve} on that edge by $d_{ve} = |\det(\Delta(M_{ve}))|$, where M_{ve} is the manifold which has plumbing diagram corresponding to the piece not containing v if one cuts $\Delta(M)$ just after v on the string corresponding to e.*

Proof. This follows since the absolute value of the determinant of the intersection matrix of a rational homology sphere graph manifold is the order of the first homology group. And that the manifold M_{ve} is the manifold corresponding to the manifold one gets by gluing in a solid torus to the boundary of the piece not containing v after cutting along the edge corresponding to e, by the gluing described above. □

Lemma 2.6. *Let v be a node in $\Gamma(M)$. Then the sign ε at v , is $\varepsilon = -\operatorname{sign}(a_{vv})$, where a_{vv} is the entry of $A(\Delta(M))^{-1}$ corresponding to the node v.*

Proof. To prove this we calculate $\widetilde{\mathrm{lk}}(L_v, L_w)$ where L_v is a nonsingular fiber at the v'th node and L_w is a non singular fiber at the w'th node. Let X be the plumbed 4-manifold given by $\Delta(M)$. Then each vertex of $\Delta(M)$ corresponds to a circle bundle over a 2-manifold in the plumbing so the i'th node gives us a 2-cycle E_i in X, and the collection of all the E_i's generate $H_2(X)$. The intersection matrix $A(\Delta(M))$ is the matrix representation for the intersection form on $H_2(X)$ in this generating set. So to construct a A_0 such that it has zero intersection with all 2-cycles, we just need that $A_0 \cdot E_i = 0$ for all i. Let D_v and D_w be transverse disk to E_v and E_w with boundaries L_v and L_w, if $v = w$ choose them disjoint. Set $A_0 = \det(\Delta(M))D_v - \sum_i \det(\Delta(M))(a_{vi})E_i$, where a_{ij} is the ij'th entry of $A(\Delta(M))^{-1}$, and choose $A_1 = D_w$. Then $A_0 \cdot E_i = 0$ for all i and $\widetilde{\mathrm{lk}}(L_v, L_w) = \frac{1}{\det(\Delta(M))} A_0 \cdot D_w = -\frac{1}{\det(\Delta(M))} \det(\Delta(M))(a_{vw})E_w \cdot D_w = -a_{vw}$, since $E_i \cdot D_w = 0$ if $i \neq w$ and $E_w \cdot D_w = 1$. □

The proof here is the same as given for proposition 9.1 in [NW08].

This way of obtaining a splice diagram shows that no edge weight on a edge to a leaf is 0, since we assumed that our plumbing diagram is in normal form, especially that all weights on strings are ≤ -2, so a weight on an edge to a leaf is

the determinant of a matrix on the form

$$\begin{pmatrix} b_{11} & -1 & 0 & \dots & 0 \\ -1 & b_{22} & -1 & \dots & 0 \\ 0 & -1 & b_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{nn} \end{pmatrix}$$

where $b_{ii} \leq 2$, and determinants of such matrices are never 0.

To prove the theorem we have to introduce another diagram, which we will call the unnormalized splice diagram $\widetilde{\Gamma}(M)$.

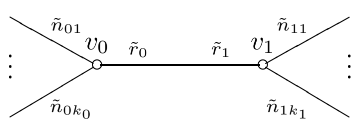
Definition 2.7. The unnormalized splice diagram $\widetilde{\Gamma}(M)$ is a tree, with the same graph structure as the splice diagram $\gamma(N)$, but it has no signs at nodes, and the weights at edges are defined to be $\tilde{d}_{ve} = \det(\Delta(M_{ve}))$.

It is clear that one constructs the unnormalized splice diagram from the plumbing diagram in the same way as the splice diagram, except that for the weight $\tilde{d}_{ev}$ on a edge e at the node v one does not take the absolute value of the $\det(\Delta(M_{ve}))$, but just sets $\tilde{d}_{ve} = \det(\Delta(M_{ve}))$, and one does not put any signs at the nodes.

Lemma 2.8. *Let $\widetilde{\Gamma}(M)$ be an unnormalized splice diagram of the rational homology sphere graph manifold M. Then $\widetilde{\Gamma}(M)$ has the same form as $\Gamma(M)$ as a graph, and for a node v we get $d_{ve} = |\tilde{d}_{ve}|$ and $\varepsilon_v = \operatorname{sign}(\Delta(M)) \prod_e \operatorname{sign}(\tilde{d}_{ve})$, where the product is taken over all edges at v.*

Proof. That the graph has the same form and $d_{ve} = |\tilde{d}_{ve}|$ is clear from the constructions. The last follows from the proof of theorem 12.2 in [NW05a]. In their theorem they assume negative definite intersection form, but if one looks at the proof one sees that for the part we need it is not necessary to assume negative definiteness. □

If we have a edge between two nodes in a unnormalized splice diagram that look likes this



Then we the define the unnormalized edge determinant $\widetilde{D}$ associated to a edge to be

$$\widetilde{D} = \tilde{r}_0 \tilde{r}_1 - \widetilde{N}_0 \widetilde{N}_1 \tag{2}$$

where $\widetilde{N}_i = \prod_{j+=}^{k_i} \tilde{n}_{ij}$. Then it's clear that $D = \operatorname{sign}(\tilde{r}_0) \operatorname{sign}(\tilde{r}_1) \widetilde{D}$.

We are also going to need what is called a maximal splice diagram, it is a tree with integer weights on edges leaving vertices.

Definition 2.9. The maximal splice diagram of a manifold M with plumbing diagram $\Delta(M)$ has the underlying graph the graph of the plumbing diagram. On edges one adds decorations as in the construction of a unnormalized splice diagram from the plumbing diagram $\delta(M)$.

To get a unnormalized splice diagram from a maximal splice diagram, one just removes the vertices of valence two and removes the decoration on edges next to vertices of valence one.

An edge in our splice diagram between nodes v_0 and v_1 corresponds to a torus T^2 which the pieces corresponding to the nodes are glued along. In that torus we get several natural knots from the Seifert fibered structure on each side, namely a fiber F_i and a section S_i from the fibration of the piece corresponding to M_i. We are going to be interested in the fiber intersection of F_0 with F_1 in the torus T^2. We make the following convention, if we write $F_0 \cdot F_1$ we mean the intersection product in T^2, where T^2 is oriented as the boundary of M_0 and when we write $F_1 \cdot F_0$ we mean the intersection product in T^2 oriented as the boundary of M_1. In this way $F_0 \cdot F_1 = F_1 \cdot F_0$, since we change the orientation on T^2 when we interchange F_0 and F_1.

Because M is a rational homology sphere, the diagram is a tree. It is then possible to orient the F_i's and S_i's such that the intersection number of the fibers F_i and F_j from the Seifert fibered pieces on each side of a separating torus is always positive, and so that $F_i \cdot S_i = 1$. It should be mentioned that S_i is only well defined up to adding a multiply of F_i, for the case of orienting the F_i's and the S_i's, it does not mater.

One does this be choosing a Seifert fibered piece corresponding to a leaf and choose a orientation on the piece. This then gives an orientation on the fiber in the boundary of that piece and we the choose the right orientation of the section. Choose a orientation on the piece glued along the torus, such that the fiber intersection number is positive, and choose the right orientation on the section. Then continue to do the same in the other boundary pieces of this second Seifert fibered piece. We can then get all the fibers and sections oriented this way, since $\Gamma(M)$ is a tree.

We will always assume our fibers and sections are oriented this way.

3. Determine the decomposition graph from a splice diagram

Given a graph 3 manifold M there is another graph invariant one can associate to the JSJ decomposition of M called the decomposition graph. We will in this section show that given the splice diagram of a manifold and the order of its first homology group, one can construct the decomposition graph of that manifold.

The decomposition graph has one node for each Seifert fibered piece of M, and a edge between nodes if they are glued by a torus. At node v one puts 2 weights, the first is the rational euler number e_v and the other number is the orbifold euler characteristic of the base χ_v^{orb}. If the Seifert fibered piece corresponding to the node v has Seifert invariant $M(g; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$ then

$$e_v = -\sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} \tag{3}$$

and

$$\chi_v^{orb} = \chi_v - \sum_{i=1}^{k} (1 - \frac{1}{\alpha_i}) \tag{4}$$

where χ_v is the euler characteristic of the base surface. This formula is in fact only true for closed Seifert fibered spaces, if the space has boundary, one needs additional information. The additional information is a simple closed curve in each of the boundary components, which we get from the JSJ decomposition, by at each piece of the boundary take a the curve corresponding to a fiber from the other side. Then one closes the manifold by gluing in solid tori in the boundary pieces, by gluing a meridian to the closed curve. Finally take $e(v)$ to be the rational euler number of this closed manifold.

One weights an edge e of the decomposition graph by the the intersection number in T of a nonsingular fibers of the Seifert fibrations on each side of the torus T corresponding to e.

One gets the graph structure of the decomposition graph of M from the splice diagram $\Gamma(M)$ of M by removing all leaves, i.e. by removing all vertices of valence one and the edges leading to them. It is clear that since nodes in the $\Gamma(M)$ corresponds to Seifert fibered pieces in the JSJ-decomposition of M, and a edge between two nodes means there are glued along a torus, that the result has the shape of the decomposition graph.

We start by given a formula for the orbifold euler characteristic

Proposition 3.1. *Let v be a node in the splice diagram $\Gamma(M)$ of the manifold M. Then*

$$\chi_v^{orb} = 2 - n(v) + \sum_e \frac{1}{d_{ve}} \tag{5}$$

where $n(v)$ is the valence of v and the sum is taken over all edges leading to leaves.

Proof. Since each leaf of the node v corresponding to the Seifert fibered piece corresponds to a singular fiber, and a singular fiber corresponds to a leaf at v, so taking the sum in (4) over singular fibers is the same as taking the sum over edges at v leading to leaves. The negative intersection matrix $-A(\Delta(M_{ve}))$ has numbers $b_i \geq 2$ on the diagonal and -1 adjacent to diagonal entries and 0 elsewhere since e leads to a leaf. We can then diagonalize $-A(\Delta(M_{ve}))$ only by adding rows and columns in the following way. If the matrix is n by n we clear the 1 at the $(n, n-1)$ entry by adding $-\frac{1}{a_{nn}}$ times the n'th row to the $n-1$'st row. Then we clear the 1 at $(n-1, n)$ by adding $-\frac{1}{a_{nn}}$ times the n'th column to the $n-1$'st column. We now have that in the n'th row and n'th column, only the diagonal entry is now zero. We then proceed to clear the $(n-1, n-2$ and $(n-2, n-1)$ entries the same way. This then continues until the matrix is diagonal.

If we do this we get that the ii'th entry of $-A(\Delta(M_{ve}))$ is $[b_i, b_{i-1}, \ldots, b_1]$ which is the continued fraction

$$[b_i, b_{i-1}, \ldots, b_1] = b_i - \cfrac{1}{b_{i-1} - \cfrac{1}{b_{i-2} - \ldots}}. \tag{6}$$

Then $d_{ve} = |\det(\Delta(M_{ve}))| = |[b_n, b_{n-1} \ldots, b_1][b_{n-1}, b_{n-2}, \ldots, b_1] \cdots [b_1]|$. The continued fraction $[b_1] = b_1$ and the denominator of $[b_i, , b_{i-1}, \ldots, b_1]$ is the numerator of $[b_{i-1}, , b_{i-2}, \ldots, b_1]$. This implies that d_{ve} is equal to the numerator of $|[b_n, , b_{n-1}, \ldots, b_1]|$. It follows from corollary 5.7 in [Neu81] that the numerator of $[b_n, b_{n-1}, \ldots, b_1]$ is equal to α, where α is the first part of the Seifert invariant of the singular fiber corresponding to the leaf at e. So $d_{ve} = \alpha$ since $\alpha > 0$. We now have that

$$\chi_v^{orb} = \chi_v - \sum_e (1 - \frac{1}{d_{ve}}) = \chi_v - l(v) + \sum_e \frac{1}{d_{ve}}. \tag{7}$$

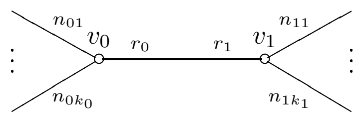
where $l(v)$ is the number of singular fibers, which is the same as the number of leaves at v. The base surface is a sphere since M is a rational homology sphere, so $\chi_v = 2 - r(v)$ where $r(v)$ is the number of boundary components which is the same as the the number of edges leading to other nodes. The formula the follows since $l(v) + r(v) = n(v)$. □

We next proves a lemma relating the fiber intersection number to the edge determinant.

Lemma 3.2 (Unnormalized edge determinant equation)**.** *Assume that we have an edge in our splice diagram between two nodes. Let T be the torus corresponding to the edge and p the intersection number in T of Seifert fibers from each of the sides of T. Let $d = \det(\Delta(M))$, then*

$$p = \frac{\widetilde{D}}{d} \tag{8}$$

Proof. Let the numbers on the edge be as in



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And let $N_i = \prod_{j=1}^{k_i} n_{ij}$ for $i \in \{0, 1\}$.

We start by proving the formula under the additional assumption that there is no edge of weight 0 adjacent to the nodes, except possibly r_0 and r_1.

Let H_i be a fiber at the i'th node. Let $L_i' \subset T^2$ be a simple curve which generates $\ker(H_1(T^2, \mathbb{Q}) \hookrightarrow H_1(M_i, \mathbb{Q}))$ where M_i is the piece of M gotten by cutting along the torus corresponding to the edge, including the piece corresponding to the node v_i. Since M is a rational homology sphere the Meyer Vietoris sequence gives us that $H_1(T^2, \mathbb{Q}) \cong H_1(M_0, \mathbb{Q}) \bigoplus H_1(M_1, \mathbb{Q})$. $H_1(M_i, \mathbb{Q}) \cong H_1(T^2, \mathbb{Q})/L_i$ by the long exact sequence and $H_1(M_i/T^2) = H_1(M/M_{i+1})$ is a finite group. This implies that L_0 and L_1 are linearly independent, so $L_0 \cdot L_1 \neq 0$, where $\cdot$ denotes the intersection product in T^2.

We have the following relation

$$a_i H_i = b_{i0} L_0 + b_{i1} L_1 \tag{9}$$

for some $a_i, b_{i0}, b_{i1} \in \mathbb{Z}$, since L_0, L_1 are linearly independent in $H^1(T^2, \mathbb{Q}) = \mathbb{Q}^2$ and hence a basis. We also note that $L_i \cdot L_i = 0$.

We now want to compute the linking numbers $\mathrm{lk}(H_i, H_j)$ for $i, j \in \{0, 1\}$. Let $C_i \subset M_i$ be such that $\partial C_i = L_i$. This implies that $a_i H_i = b_{i0} \partial C_0 + b_{i1} \partial C_1$. Then one can compute $\mathrm{lk}(H_i, H_j)$ as $\mathrm{lk}(H_i, \frac{1}{a_j}(b_{j0} \partial C_0 + b_{j1} \partial C_1)$, but this is the same as to compute $H_i \bullet (\frac{1}{a_j}(b_{j0} C_0 + b_{j1} C_1)$, where $\bullet$ denotes the intersection number in M. Now C_0 lives in the M_0 piece and C_1 in the M_1 piece, so when one computes $H_0 \bullet (\frac{1}{a_j}(b_{j0} C_0 + b_{j1} \frac{1}{c_1} C_1)$, it is only the C_0 parts that matters, since H_0 is in the M_0 piece, and therefore does not intersect things in the M_1 piece. This means we can compute $\mathrm{lk}(H_0, H_j)$ as $H_0 \bullet (\frac{1}{a_j} b_{j0} C_0)$.

T^2 has a collar neighborhood in M_0, so when we want to compute $\mathrm{lk}(H_0, H_0)$ we can assume that the push-off of one of the copies of H_0 in T^2 lives in this collar neighborhood. I.e. if the collar neighborhood is $(0, 1] \times T^2$, then the push off is $s \times H_0$ for some $s \in (0, 1]$. Over the collar neighborhood C_0 is just $(0, 1] \times L_0$, so

$$\begin{aligned} H_0 \bullet (\frac{1}{a_0} b_{00} C_0) &= (s \times H_0) \bullet (\frac{1}{a_0} b_{00}((0, 1] \times c_0 L_0)) \\ &= H_0 \cdot (\frac{1}{a_0} b_{00} L_0) \\ &= \frac{1}{a_0}(b_{00} L_0 + b_{01} L_1) \cdot (\frac{1}{a_0} b_{00} L_0) \\ &= \frac{1}{a_0^2} b_{01} b_{00} (L_1 \cdot L_0). \end{aligned}$$

So we get that $\mathrm{lk}(H_0, H_0) = \frac{1}{a_0^2} b_{01} b_{00} (L_1 \cdot L_0)$. By a similar calculation one gets that $\mathrm{lk}(H_0, H_1) = \frac{1}{a_0 a_1} b_{01} b_{10} (L_1 \cdot L_0)$ and $\mathrm{lk}(H_1, H_1) = \frac{1}{a_1^2} b_{10} b_{00} (L_1 \cdot L_0)$.

Another way to calculate the linking number of two fibers is that it's given by the inverse intersection matrix. More precisely, the linking number of a fiber at the i'th piece in a plumbing diagram $\Delta(M)$ of M with a fiber at the j'th piece is given by the negative of the (i,j)'th entry of $A(\Delta(M))^{-1}$, where $A(\Delta(M))$ is the intersection matrix of the plumbing $\Delta(M)$, as we showed in the proof of Lemma 2.6. By theorem 12.2 in [NW05a] it is equal to $\frac{l_{ij}}{\det(\Delta(M))}$, where l_{ij} is the product of the weights adjacent to but not on the path from the i'th node to the j'th node in the splice diagram and n is the number of vertices in $\Delta(M)$. In their theorem they are calculating l_{ij} in a maximal splice diagram, but it is clear that if one has calculated the maximal splice diagram from the plumbing diagram $\Delta(M)$, one gets our unnormalized splice diagram from the maximal one, by removing any vertices with only 2 edges and not changing any weights. So if i and j represents Seifert fibered pieces, then one gets the same number l_{ij} by calculating it in our unnormalized splice diagram, since no vertices with only 2 edges can contribute to l_{ij}.

Returning to our situation, we then get the the following equations for the linking numbers using the notation from above. $\mathrm{lk}(H_0,H_0)=\frac{N_0r_0}{d}$, $\mathrm{lk}(H_1,H_1)=\frac{N_1r_1}{d}$ and $\mathrm{lk}(H_0,H_1)=\frac{N_0N_1}{d}$. Combining this with our other equations for the linking numbers we get.

$$\frac{N_0r_0}{d}=\frac{1}{a_0^2}b_{01}b_{00}(L_1\cdot L_0) \tag{10}$$

$$\frac{N_1r_1}{d}=\frac{1}{a_1^2}b_{10}b_{11}(L_1\cdot L_0) \tag{11}$$

$$\frac{N_0N_1}{d}=\frac{1}{a_0a_1}b_{01}b_{10}(L_1\cdot L_0) \tag{12}$$

It follows from (12) that the $b_{ij}\neq 0$, since $N_i\neq 0$ by our assumptions. So we can divide the product of (10) and (11) by (12), this gives us.

$$\frac{r_0r_1}{d}=\frac{1}{a_0a_1}b_{00}b_{11}(L_1\cdot L_0) \tag{13}$$

Let us now compute p, which is equal to $H_0\cdot H_1$ by definition,

$$\begin{aligned}H_0\cdot H_1 &= \frac{1}{a_0}(b_{00}L_0+b_{01}L_1)\cdot\frac{1}{a_1}(b_{10}L_0+b_{11}L_1)\\ &= \frac{1}{a_0a_1}(b_{01}b_{10}L_1L_0+b_{00}b_{11}L_0L_1)\\ &= \frac{r_0r_1-N_0N_1}{d}\\ &= \frac{\widetilde{D}}{d}\end{aligned}$$

Here we use (12), (13) and the definition of $\widetilde{D}$.

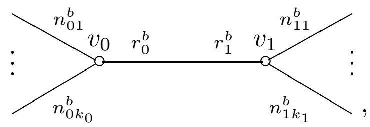
We have now proved the the equality

$$dp=r_0r_1-\prod_{i=1}^{k_0}n_{0i}\prod_{j=1}^{k_1}n_{1j} \tag{14}$$

whenever $n_{ij}\neq 0$. Now this is a equation concerning minors of the negative intersection matrix $-A(\Delta(M))$ of M. We want to see what happens if we vary the diagonal entries of $-A(\Delta(M))$. We are especially interested in what happens when we change the entries of the diagonal corresponding to changing one of the n_{ij}'s. Let a be a entry on the diagonal of $-A(\Delta(M))$ which lies in the minor n_{0l}. Replacing a with any integer b, we get a new matrix $-A(\Delta(M_b))$, which is the intersection

matrix of the graph manifold M_b one gets from a plumbing diagram corresponding to one for M with the weight corresponding to a replaced with b. For all values of of b, except maybe one, M_b is a rational homology sphere, since by computing $d = \det(\Delta(M_b))$ by expanding by the row which include b, one gets $d = bA + B$ which has at most one solution for $d = 0$, because $aA + B \neq 0$.

M_b has splice diagram with the same form as the one for M, in particular around the node we are working with it look like



The only weights of the splice diagram of M_b which are different of the weights from the splice diagram of M, are n_{0l} and r_1 since none of the others see the the entry of $-\Delta M_b$ which we have changed. Again $n_{0l}^b = bA_{01} + B_{01}$ and $n_{01} = aA_{01} + B_{01}$, so $n_{0l}^b = 0$ for at most one value of b. So for all but maybe two values of b, we have the following equation

$$d_b p = r_0 r_1^b - n_{0l}^b \prod_{\substack{i=1 \\ i \neq l}}^{k_0} n_{0i} \prod_{j=1}^{k_1} n_{1j}. \tag{15}$$

Let $\widetilde{N}_l = \prod_{\substack{i=1 \\ i\neq l}}^{k_0} n_{0i} \prod_{j=1}^{k_1} n_{1j}$. We get that $r_1^b = bA_1 + B_1$ and the above equation becomes

$$(bA + B)p = r_0(bA_1 + B_1) - (bA_{01} + B_{01})\widetilde{N}_l. \tag{16}$$

This is equivalent to

$$b(Ap - A_1 r_0 + A_{01}\widetilde{N}_j) = Bp - B_1 r_0 + B_{01}\widetilde{N}_j. \tag{17}$$

Since this is true for more than one value of b, it implies that

$$Ap - A_1 r_0 + A_{01}\widetilde{N}_j = Bp - B_1 r_0 + B_{01}\widetilde{N}_j = 0. \tag{18}$$

But this implies that equation (17) holds for any value of b. So the equation $dp = D$ holds even if we change the diagonal entries of $-\Delta(M)$, so, in particular, it holds if some $n_{ij} = 0$. Now since we are only interested in rational homology spheres, and for them $d \neq 0$, we get that the unnormalized edge determinant equation always holds, by dividing by d.

□

We get following corollary, just by using the relation between D and $\widetilde{D}$, that $|\det(\Delta(M))| = |H_1(M)|$ and taking absolute value.

Corollary 3.3 (Edge determinant equation). *For an edge between nodes in the splice diagram for the rational homology sphere graph manifold M, we get the following equation*

$$p = \frac{|D|}{|H_1(M)|} \tag{19}$$

where p is the intersection number in the torus corresponding to the edge of a fiber from each of the sides of the torus, and D is the edge determinant associated to that edge.

A consequence of the edge determinant equation is that no node in the splice diagram can have more than one adjacent edge weight of value 0. This is because we know that no leaf has edge weight 0, so if we have a node with at least two adjacent edge weights of value 0, the edge determinant of an edge with 0 on would

be $0r_1 - \varepsilon_0\varepsilon_1 0 N_1 = 0$, and then the edge determinant equation implies that $p = 0$. But $p = 0$ means that the fibers from each side of the torus corresponding to the edge have intersection number 0, but then we could extend the fibration over T^2. So the nodes v_0 and v_1 correspond to one node v corresponding to a Seifert fibered piece, which would not be cut in the JSJ decomposition.

Next we need a formula for computing the rational euler class of the Seifert fibered pieces of our graph manifold, using only information from the splice diagram and the order of the first homology group. If we have a node in our splice diagram, as in Fig. 1 below, where everything to the left is leaves

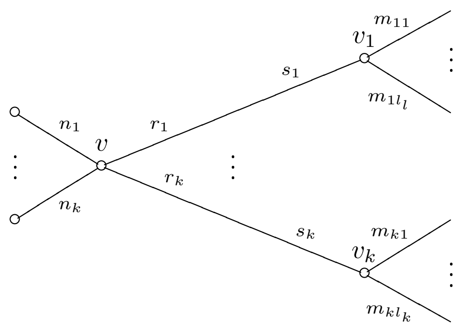


Figure 1 ,

we let $N = \prod_{j=1}^{k} n_j$ and let $M_i = \prod_{j=1}^{l_i} m_{ij}$. Then we have the following proposition.

Proposition 3.4. *Let v be a node in a splice diagram decorated as in Fig. 1 above with $r_i \neq 0$ for $i \neq 1$, let e_v be the rational euler number of the Seifert fibered piece corresponding to v, then*

$$e_v = -d\Big(\frac{\varepsilon s_1}{N D_1 \prod_{j=2}^{k} r_k} + \sum_{i=2}^{k} \frac{\varepsilon_i M_i}{r_i D_i}\Big) \tag{20}$$

where $d = |H_1(M)|$ and D_i is the edge determinant associated to the edge between v and v_i.

Proof. We start by proving a formula for $e(v)$ using an unnormalized splice diagram, and then show that the relation between unnormalized and normalized splice diagram will give us the formula.

We first assume that $r_1 \neq 0$ and prove the formula under that hypothesis.

Let $\Gamma(M)$ be a non normalized splice diagram, looking like the above one. It is constructed from the plumbing diagram Δ, which look like Fig. 2 below around the node v.

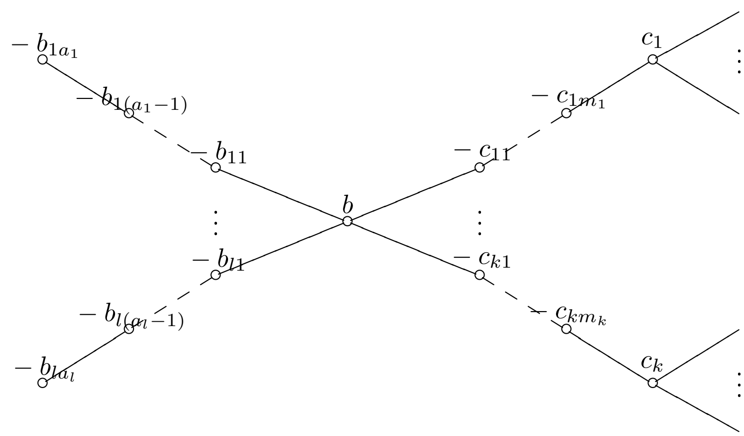


Figure 2 ,

where $b_{ij}, c_{ij} \geq 2$.

We want to compute $\det(-\Delta)$, so we look at the negative intersection matrix $-A(\Delta)$ of Δ, which we can write like

$$-A(\Delta) = \begin{pmatrix} b & -1 & 0 & \dots & -1 & \dots & -1 & \dots & -1 & \dots \\ -1 & b_{11} & -1 & \dots & 0 & \dots & 0 & \dots & 0 & \dots \\ 0 & -1 & b_{12} & \dots & 0 & \dots & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots & \\ -1 & 0 & 0 & \dots & b_{21} & \dots & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots & \\ -1 & 0 & 0 & \dots & 0 & \dots & c_{11} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots & \\ -1 & 0 & 0 & \dots & 0 & \dots & 0 & \dots & c_{21} & \dots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \ddots \end{pmatrix}$$

We get it the following way. If we delete the b-weighted vertex $v \in \Delta$ we get l components on the left and k components on the right.

If we let B_i be the negative intersection matrix of the i'th component to the left. It is of the form

$$B_i = \begin{pmatrix} b_{i1} & -1 & \dots & 0 & 0 \\ -1 & b_{i2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{ia_{i-1}} & -1 \\ 0 & 0 & \dots & -1 & b_{ia_i} \end{pmatrix}.$$

Likewise we let C_i be the negative intersection matrix of the i'th component to the right.

$$C_i = \begin{pmatrix} c_{i1} & -1 & \dots & 0 & 0 & \dots \\ -1 & c_{i2} & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & c_{im_i} & -1 & 0 \\ 0 & 0 & \dots & -1 & c_i & -1 \\ \vdots & \vdots & \dots & 0 & -1 & \ddots \end{pmatrix}$$

We get that $-A(\Delta)$ has b at the A_{11} entry, then the B_i's and the C_i's are following along the diagonal. The first row and column has a -1 in the column/row there corresponds to the upper left corner of a B_i or C_i, and all other entries 0.

$$-A(\Delta) = \begin{pmatrix} b & -1 & & -1 & -1 & & -1 \\ -1 & B_1 & & & & & \\ & & \ddots & & & & \\ -1 & & & B_l & & & \\ -1 & & & & C_1 & & \\ & & & & & \ddots & \\ -1 & & & & & & C_k \end{pmatrix}$$

We will diagonalize the matrix to compute $\det(-\Delta) = \det(-A(\Delta))$. This can be done by first diagonalising the matrix, except for the first row and column. To see how this is done we look at one of the B_i. We clear the off-diagonal term in the last column by adding $\frac{1}{b_{ia_i}}$ times the last row from the second to last row, then we can clear the -1 at the left of the b_{ia_i} by a symmetric argument. We have now cleared the off diagonal terms of the last row and column, and at the second to last diagonal entry we have $b_{ia_{i-1}} - \frac{1}{b_{ia_1}}$. Since all the $b_{ij} \geq 2$ we can continue doing this using the last row and columns with off-diagonal entries to clear the one before it. By diagonalising B_i this way we only add rows and columns, and we never use the first row and column of B_i, this assures us that it does not change the matrix outside the B_i block, since the rows and columns we use have zeros outside B_i. The first entry of the block after diagonalising it will then be

$$\beta_i = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{b_{i3} - \dots}}$$

We can also diagonalise the C_i's in the same way, by starting at the bottom right corner and working up, only adding rows and columns that are not the first row and column. We will denote the first entry of the diagonalization of C_i by γ_i

To get the matrix completely diagonal we have to remove the -1 in the first row and the first column. If -1 is in the first row corresponds to the entry β_i of a diagonalized B_i then we remove it by adding $\frac{1}{\beta_i}$ times the i'th row to the first. This changes the the first entry by subtracting $\frac{1}{\beta_i}$. Similarly if the -1 corresponds to the entry γ_i of the diagonalized C_i we let

$$\lambda_i = \gamma_i - c_{i1} - \cfrac{1}{c_{i2} - \cfrac{1}{c_{i3} - \dots}}$$

Our assumptions on the splice diagram assures the $A_{ii} \neq 0$ since $\det(C_i) = r_i$ and $A_{ii} \mid \det(C_i)$. We let

$$\xi_i = \cfrac{1}{c_{i1} - \cfrac{1}{c_{i2} - \cfrac{1}{c_{i3} - \dots}}} - \cfrac{1}{c_{i1} - \cfrac{1}{c_{i2} - \cfrac{1}{c_{i3} - \dots}} + \lambda_i}$$

So the change to the first entry of the matrix will be adding

$$\xi_i - \cfrac{1}{c_{i1} - \cfrac{1}{c_{i2} - \cfrac{1}{c_{i3} - \dots}}}$$

We then get that first entry of the diagonalised matrix is

$$b - \sum_{i=1}^{l} \cfrac{1}{b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{b_{i3} - \dots}}} - \sum_{i=1}^{k} \cfrac{1}{c_{i1} - \cfrac{1}{c_{i2} - \cfrac{1}{c_{i3} - \dots}}} + \sum_{i=1}^{k} \xi_i$$

Now we know that

$$-e_v = b - \sum_{i=1}^{l} \cfrac{1}{b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{b_{i3} - \dots}}} - \sum_{i=1}^{k} \cfrac{1}{c_{i1} - \cfrac{1}{c_{i2} - \cfrac{1}{c_{i3} - \dots}}}$$

by arguments of Walter Neumann in the proof of theorem 3.1 in [Neu97]. So if $d = \det(A(\Delta))$ we get that

$$d = (-e_v + \sum_{i=1}^{k} \xi_i) \prod_{i=1}^{l} \det(B_1) \prod_{i=1}^{k} \det(C_i) \tag{21}$$

But we also know that $n_i = \det(B_i)$ and $r_i = \det(C_i)$, so we get the following formula.

$$d = (-e_v + \sum_{i=1}^{k} \xi_i) \prod_{i=1}^{l} n_i \prod_{i=1}^{k} r_i = (-e(v) + \sum_{i=1}^{k} \xi_i) N \prod_{i=1}^{k} r_i \tag{22}$$

If we cut M along the torus just before the Seifert fibered piece corresponding to the node c_j, when coming from v, and glue in a solid tori, we get a graph manifold with a non normalized splice diagram Γ' corresponding to Γ, where we remove everything after r_j, so that r_j becomes the weight corresponding to a leaf. And the plumbing diagram Δ' of this manifold corresponds to Δ with everything after c_{jm_j} removed. We can now make the same calculation $\det(\Delta')$ as above and get that

$$\det(\Delta') = p_j(-e_v + \sum_{\substack{i=1\\ i\neq j}}^{k} \xi_i) N \prod_{\substack{i=1\\ i\neq j}}^{k} r_i \tag{23}$$

where

$$p_j = \det \begin{pmatrix} c_{i1} & -1 & \dots & 0 \\ -1 & c_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{im_i} \end{pmatrix}$$

But it follows from the proof of theorem 3.1 in [Neu97] that p_j is the fiber intersection number for the edge. By definition $\det(\Delta') = s_j$. So by combining (22) and (23) we get that

$$-\xi_j = \frac{s_j}{p_j N \prod_{\substack{i=1\\ i\neq j}}^{k} r_i} - \frac{d}{N \prod_{i=1}^{k} r_i} = \frac{s_j r_j - d p_j}{p_j N \prod_{i=1}^{k} r_i}$$

by using that $p_j = \frac{\widetilde{D}}{d}$ by 3.2 we get

$$-\xi_j = d\frac{s_j r_j - \widetilde{D}_j}{\widetilde{D}_j N \prod_{i=1}^k r_i} = d\frac{N M_j \prod_{\substack{i=1 \\ i\neq j}}^k r_i}{\widetilde{D}_j N \prod_{i=1}^k r_i} = \frac{dM_j}{r_j \widetilde{D}_j}$$

So using this in (22) we get

$$e(v) = -d\left(\frac{1}{N\prod_{j=1}^k r_k} + \sum_{i=1}^k \frac{M_i}{r_i\widetilde{D}_i}\right) \tag{24}$$

$$= -d\left(\frac{\widetilde{D}_1}{N\widetilde{D}_1\prod_{j=1}^k r_k} + \sum_{i=1}^k \frac{M_i}{r_i\widetilde{D}_i}\right) \tag{25}$$

$$= -d\left(\frac{r_1 s_1 - N\prod_{j=1}^k r_k M_1}{N\widetilde{D}_1\prod_{j=1}^k r_k} + \sum_{i=1}^k \frac{M_i}{r_i\widetilde{D}_i}\right) \tag{26}$$

$$= -d\left(\frac{s_1}{N\widetilde{D}_1\prod_{j=2}^k r_k} + \sum_{i=2}^k \frac{M_i}{r_i\widetilde{D}_i}\right) \tag{27}$$

This proves the formula if $r_1 \neq 0$. We get a new equation by multiplying both sides of (27) by $\prod_{i=1}^k \widetilde{D}_i \prod_{i=2}^k r_k$. This equation is as in the proof of unnormalized edge determinant equation, an equation in the minors of $-\Delta(M)$. By changing a diagonal entry b of $-\Delta(M)$, lying in C_1, we get that the equation becomes a polynomial equation in b, which is an equality for infinity many values of b, hence it is an equality, so the equation holds for all value of b. We get our formula by dividing this equation by $\prod_{i=1}^k \widetilde{D}_i \prod_{i=2}^k r_i$, which is not 0 by our assumption on the r_i's.

We saw earlier that $\varepsilon = \operatorname{sign}(d)\prod_{i=1}^l \operatorname{sign}(n_i)\prod_{i=1}^k \operatorname{sign}(r_i)$, we get that $\frac{d}{N\prod_{i=1}^k r_i} = \frac{\varepsilon|d|}{|N|\prod_{i=1}^k |r_i|}$.

Using that $D_i = \operatorname{sign}(r_i)\operatorname{sign}(s_i)\widetilde{D}$ and $\varepsilon_i = \operatorname{sign}(M_i)\operatorname{sign}(s_i)\operatorname{sign}(d)$, we also get that $d\frac{M_i}{r_i\widetilde{D}_i} = |d|\frac{\varepsilon_j|M_i|}{|r_i|D_i}$, which proves the proposition.

□

From Corollary 3.3 and Propositions 3.1 and 3.4 we get the information needed to make the decomposition graph.

4. Proof of the First Main Theorem

In last section we saw that knowing the splice diagram and the order of the first homology group lets us construct the decomposition graph of M. It therefore also lets us construct the decomposition matrix, also called the reduced plumbing matrix as defined in [Neu97]. By theorem 3.1 in [Neu97] we just need to show that the decomposition matrix is negative definite if all edge determinants are positive and the splice diagram has no negative decorations.

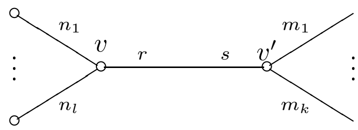
Theorem 4.1. *Let M be a rational homology sphere graph manifold, with splice diagram Γ, such that all edge determinants are positive and Γ has no negative decorations at nodes. Then M is a singularity link.*

Proof. The assumption that all edge determinants are > 0 and we do not have any negative decorations at edges assures that no edge weight is 0. Because if we had an edge weight of 0, then it had to be on a edge between nodes, but the edge determinant of this edge would be $0r_1 - \varepsilon_0\varepsilon_1 N_0 N_1 = -N_0N_1 < 0$.

Let $d = |H_1(M)|$. We proceed by induction in the number of nodes of Γ. If Γ only has one node, then M is Seifert fibered and the reduced plumbing matrix is a

1×1 matrix, with the rational euler number e of M as it's entry. By Proposition 3.4, $e = -d\frac{\varepsilon}{N\prod_{j=0}^{k} r_k}$. But N, r_k, d are all greater than 0 by definition, and $\varepsilon = 1$ by assumption, so e is negative. Hence the reduced plumbing matrix is negative definite.

Assume that there are n nodes in Γ. Let v be a end node of Γ, meaning a node of the form



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such nodes always exist since Γ is a tree. Then the reduced plumbing matrix is of the form

$$\begin{pmatrix} e(v) & \frac{1}{p} & 0 & \dots \\ \frac{1}{p} & e(v') & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix}$$

If we set $N = \prod_{i=0}^{l} n_i$ and $M = \prod_{i=0}^{k} m_i$ we get by Proposition 3.4 that

$$e(v) = -d\frac{\varepsilon s}{DN}$$

where D is the edge determinant of the edge between v and v'. This means that the matrix look like this

$$\begin{pmatrix} \frac{-\varepsilon sd}{DN} & \frac{1}{p} & 0 & \dots \\ \frac{1}{p} & e(v') & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix}$$

By a row and column operation we get the matrix to the form

$$\begin{pmatrix} \frac{-\varepsilon sd}{DN} & 0 & 0 & \dots \\ 0 & e(v') + \frac{\varepsilon DN}{p^2 sd} & & \\ 0 & & \ddots & \\ \vdots & & & \end{pmatrix} = \left(\tfrac{-\varepsilon sd}{DN}\right) \oplus \begin{pmatrix} e(v') + \frac{\varepsilon Nd}{Ds} & \\ & \ddots \\ & \end{pmatrix},$$

where the equality follows from using that $\frac{1}{p^2} = \frac{d^2}{D^2}$. Since s, d, N are positive by definition and D, ε are positive by assumption, the reduced plumbing matrix is negative definite if the matrix

$$\begin{pmatrix} e(v') + \frac{\varepsilon Nd}{Ds} & \\ & \ddots \\ & \end{pmatrix}$$

is negative definite. Now

$$e(v') + \frac{\varepsilon Nd}{Ds} = -\frac{d}{Ms} - \sum_{i=0}^{k} \frac{\varepsilon_i M_i}{r_i D_i} - \frac{\varepsilon Nd}{Ds} + \frac{\varepsilon Nd}{Ds} = -\frac{d}{Ms} - \sum_{i=0}^{k} \frac{\varepsilon_i M_i}{r_i D_i} = \tilde{e}(v')$$

But $\tilde{e}(v')$ is the rational euler number of the Seifert fibered piece corresponding to v' in the manifold M' which is the manifold one gets by cutting M along the edge

between v and v', and gluing in a solid tori in the piece containing v'. Then the matrix

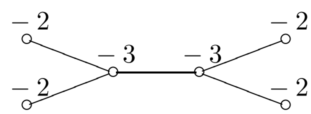
$$\begin{pmatrix} \tilde{e}(v') & & \\ & \ddots & \\ & & \end{pmatrix}$$

is the reduced plumbing matrix for the manifold M'. But since the splice diagram of M' is the same as Γ except it has a leaf instead of the node v, it only has $n-1$ nodes. Then by induction the reduced plumbing matrix of M' is negative definite, so the reduced plumbing matrix of M is negative definite, and then by theorem 3.1 in [Neu97] M is the link of a complex surface singularity.

□

5. Graph orbifolds

To prove the second main theorem we need to extend our notions and results about graph manifolds. The reason is, that in the proof we are doing induction on our graph manifold, which means we have to cut our manifold along a torus and glue in solid tori to get some smaller manifolds in which the statement holds by induction and whose universal abelian cover contribute pieces to the universal abelian cover of M. The problem is that we do not always get manifolds when we glue in the solid torus. Already in the simple case with the following plumbing diagram,



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the spaces one gets by gluing in the solid tori if one cuts along the central edge are not manifolds.

Fortunately the space we get when glue in the solid torus is not that bad, and is what we will call a graph orbifold, which we define as follows.

Definition 5.1. Let M be a 3 dimensional orbifold. We call M a graph orbifold if there exist a collecting of disjoint smoothly embedded tori $T_i \subset M$, such that each connected component of $M - \bigcup T_i$ is an S^1 orbifold bundle over orbifold surfaces.

It is clear that if a connected component of $M - \bigcup T_i$ is smooth, then it is a Seifert fibered manifold, hence if M is smooth it is a graph manifold.

We want to define the splice diagram of a rational homology graph orbifold. But to do this we have to consider which homology we are going to use. Remember that if M is smooth then $\pi_1^{orb}(M) = \pi_1(M)$ where $\pi_1^{orb}(M)$ is the orbifold fundamental group defined by Thurston see e.g. [Sco83]. So in the case of smooth manifolds orbifold coverings and coverings are the same. We need orbifold coverings, and the interesting homology group is then $H_1^{orb}(M)$, which for our purpose it is enough to define as the abelianization of $\pi_1^{orb}(M)$, since it governs the abelian orbifold covers of M. It should be mentioned that there exists a de Rham theorem for orbifold cohomology with rational coefficients, which says that $H^*_{orb}(X;\mathbb{Q}) \cong H^*(X;\mathbb{Q})$. So an orbifold is a rational homology sphere as an orbifold if and only if its underlying space is a rational homology sphere. Orbifolds also satisfies Poincare duality with rational coefficients. See e.g. [ALR07] for these results.

Next we look at the decomposition of a graph orbifold M into fibered pieces. To have a unique decomposition we do it the following way. Start by removing a solid torus neighborhood of each orbifold curve $K_j \subset N_j \subset M$, i.e. a curve along which M is not a manifold. Let $M' = M - \bigcup_{j=1}^m N_j$, then M' is a graph manifold with

m torus boundary components. We then take the JSJ decomposition of M', and glue the N_j's back in the pieces of the JSJ decomposition of M. This give us our decomposition of M into fibered pieces. It is unique since the JSJ decomposition of M' is unique.

To define the splice diagram $\Gamma(M)$ of a graph orbifold M, we start by taking a node for each connected component of $M - \bigcup_{i=1}^{n} T_i$, where the set of T_i comes from the decomposition we defined above. We then connect two nodes in $\Gamma(M)$ if the corresponding connected components of $M - \bigcup_{i=1}^{n} T_i$ are glued along a torus. We add a leaf at a node for each singular fiber of the S^1 orbifold bundle over the orbifold surface Σ, this is the same as adding a leaf for each point in Σ which does not have trivial isotropy group.

To put decorations on $\Gamma(M)$ we do the same as in the splice diagram, except that where for a manifold we used the first singular homology group, we now use the first orbifold homology group. That is, to get decorations on a edge we cut M along the corresponding torus, glue in a solid torus in the same way as for manifolds and take the order of the first orbifold homology group of the new graph orbifold as the decoration, and at nodes we put the sign of the linking number of two non singular fibers of the S^1 fibration corresponding to the node.

Let us take a closer look at the orbifold curves. In any 3 dimensional orbifold M, an orbifold curve $K \subset M$ is a embedding of S^1 such that there exist a neighborhood N_K of K and $N_K - K$ is smooth. Now N_K can be chosen to be topological a solid torus, and in this case $H_1^{orb}(N_K) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. We call p the orbifold degree of K. Another way to view N_K is as a S^1 fibration over a disk D_α with one orbifold point where the isotropy group is $\mathbb{Z}/\alpha\mathbb{Z}$. Then N_K is defined by a integer β which tells you how much the fibers over the non orbifolds points wrap around the singular fiber over the orbifold point. Now if $\gcd(\alpha, \beta) = 1$ then N_K is in fact smooth, and K is not a orbifold curve, but a singular fiber of the Seifert fibration in a neighborhood of K. In general a calculation shows that if K is a orbifold curve of degree p, then $\gcd(\alpha, \beta) = p$.

Next look at how N_K is glued to $M' = \overline{M - N_K}$. We have a collar neighborhood $U = (0, 1] \times T^2$ of $\partial(M')$. The fibration of $N_K \to D_\alpha$ gives a fibration $\partial N_K \to S^1$, this again gives a fibration of ∂U which can be extended to all of U. The image of a meridian N_K in ∂U defines a simple closed curve transverse to the fibration. By a meridian of N_K we mean a simple closed curve of the boundary, that is transverse to the fibration and has homology class of finite order in $H_1^{orb}(N_K)$. The fibration on U and the simple closed curve transverse to the boundary uniquely describe a way to glue in a solid torus in the boundary of U to make it an Seifert fibered manifold, we call this manifold M_K. Note that the gluing maps $\varphi \colon \partial M' \to \partial N_K$ and $\varphi' \colon \partial M' \to \partial(S^1 \times D^2)$ are the same, and it is therefore also clear that as topological spaces M and M_K are the same.

Proposition 5.2. *Let $K \subset M$ be an orbifold curve of degree p in a rational homology sphere orbifold M. Then $|H_1^{orb}(M)| = p|H_1^{orb}(M_K)|$.*

Proof. We get the following exact sequence from the Meyer-Vietoris sequence of the cover of M by M' and N_K

$$0 \to \mathbb{Z}^2 \xrightarrow{i_*} H_1^{orb}(M') \oplus \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to H_1^{orb}(M) \to 0 \tag{28}$$

by using that $H_1^{orb}(N_K) = \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. The zero follows since M is a rational homology sphere, hence $H_2^{orb}(M)$ has to be finite, and therefore have image zero in $\mathbb{Z}^2$. We likewise get the exact sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{i'_*} H_1^{orb}(M') \oplus \mathbb{Z} \to H_1^{orb}(M_K) \to 0 \tag{29}$$

from the Meyer-Vietoris sequence of M_K by the cover of M' and $S^1 \times D^2$. Now $i_* = (i'_*, g)$ where the image of g is in $\{0\} \times \mathbb{Z}/p\mathbb{Z}$ from the way we constructed M_K above. We have the following maps $\pi\colon H_1^{orb}(M') \oplus \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to H_1^{orb}(M') \oplus \mathbb{Z}$ given by $\pi(a,b,c) = (a,b)$ and $f_*\colon H_1^{orb}(M) \to H_1^{orb}(M_K)$ which is the map induced on orbifold homology groups, by the homeomorphism $f\colon M \to M_K$ which is the identity on the compliment of N_K. Note that $f|_{M'}\colon M' \to M'$ is the identity and $f|_{N_K}\colon N_K \to S^1 \times D^2$ induces the map from $\mathbb{Z} \bigoplus \mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}$ given by $(b,c) = b$, so we have the following map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "i_*"] \arrow[d, "\cong"] & H_1^{orb}(M') \oplus \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "\pi"] & H_1^{orb}(M) \arrow[r] \arrow[d, "f_*"] & 0 \\ 0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "i'_*"] & H_1^{orb}(M') \oplus \mathbb{Z} \arrow[r] & H_1^{orb}(M_K) \arrow[r] & 0 \end{tikzcd} \tag{30}$$

Using the snake lemma on (30) we get the following short exact sequence.

$$0 \to \mathbb{Z}/p\mathbb{Z} \to H_1^{orb}(M) \to H_1^{orb}(M_K) \to 0 \tag{31}$$

and since this is a short exact sequence of finite abelian groups, the order of the group in the middle is the product of the order of the other two groups, which proves the theorem. □

Corollary 5.3. *Let M be a rational homology sphere graph orbifold and $\overline{M}$ be its underlying topological manifold then $|H_1^{orb}(M)| = P|H_1(\overline{M})|$, where P is the product of the degrees of all orbifold curves in M.*

We say that a edge weight r of a splice diagram sees a leaf v of that splice diagram if, when we delete the node which r is adjacent to, the leaf v and the edge which r is on are in the same connected component.

Corollary 5.4. *The splice diagram $\Gamma(M)$ is equal to the splice diagram $\Gamma(\overline{M})$ except if a edge weight sees a leaf corresponding to an orbifold curve of M it is multiplied by the degree of the orbifold curve.*

Proof. If an edge weight r sees a leaf then the orbifold curve corresponding to that leaf is in the orbifold piece whose order of the first homology group gives r. □

Corollary 5.5. *Assume that we have an edge in our $\Gamma(M)$ between two nodes. Let T be the torus corresponding to the edge and p the intersection number in T of non-singular fibers from each of the sides of T. Let $d = |H_1^{orb}(M)|$, then*

$$p = \frac{|D|}{d} \tag{32}$$

where D is the edge determinant of that edge.

Proof. Since T is in the smooth part of M, the fiber intersection number is the same in M and M', and hence the same in $\overline{M}$. The equation holds in $\overline{M}$ by 3.3, and, since each term of D sees each orbifold curve once, it holds in M. □

Corollary 5.6. *Let v be a node in a splice diagram decorated as in Fig. 1 with $r_i \neq 0$ for $i \neq 1$, let $e(v)$ be the rational euler number of the S^1 fibered orbifold piece corresponding to v, then*

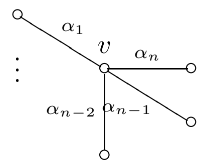
$$e(v) = -d\Big(\frac{\varepsilon s_1}{ND_1 \prod_{j=2}^k r_k} + \sum_{i=2}^k \frac{\varepsilon_i M_i}{r_i D_i}\Big) \tag{33}$$

where $d = |H_1^{orb}(M)|$ and D_i is the edge determinant associated to the edge between v and v_i.

Proof. Since the rational euler number associated to the node is the same in M and $\overline{M}$ and the formula holds for $\overline{M}$ by 3.4, it follows by noticing that the same orbifold degrees show up in the numerator and the denominator. □

6. Proof of the Second Main Theorem

A splice diagram for a Seifert fibered manifold M, which has Seifert invariant $(0;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ look like this



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and by 3.4 the sign of the rational euler number is equal to minus the sign at the node of the splice diagram. So from the splice diagram, we can read off the α_i's and the sign of the rational euler number, but this is exactly the information that determines the universal abelian cover of M. By theorem 8.2 in [Neu83a] the universal abelian cover of M is homeomorphic to the Brieskorn complete intersection $\sum(\alpha_1,\dots,\alpha_n)$ provided $e<0$. If $e>0$ one has to compose with a orientation reversing map. The case $e=0$ does not occur for a rational homology Seifert fibered manifold. We will generalize this this result to graph manifolds. But to do this we need to prove it for graph orbifolds.

A S^1 fibered orbifold will have a splice diagram as above in the case of Seifert fibered manifolds, and its universal abelian cover will likewise be $\sum(\alpha_1,\dots,\alpha_n)$, but now it follows from the proof of the above theorem in [Neu83b] which also works when $\gcd(\alpha_i,\beta_i)\neq 1$.

Now this will prove the induction start in most cases but to prove it in general we need the following lemma.

Lemma 6.1. *Let $\pi_1\colon M\to M_1$ and $\pi_2\colon M\to M_2$ be universal abelian orbifold covers such that $\deg(\pi_1)=\deg(\pi_2)=d$ and both M_1 and M_2 have an orbifold curve of degree p. Let $L(n,m_1)$ and $L(n,m_2)$ be orbifold quotients of S^3 by $\mathbb{Z}/n\mathbb{Z}$ which contain orbifold curves of degree p. Then the universal abelian cover of $L(n,m_1)\#_pM_1$ is homeomorphic to $L(n,m_2)\#_pM_2$, where $\#_p$ means taking connected sum along B^3 that intersects the orbifold curve of degree p, and the degree of the cover is nd/p.*

Proof. We are going to prove the lemma by constructing the universal abelian cover of $L(n,m_i)\#_pM_i$, and see it is determined by M,n,d and p.

Let $B^2_p\subset M_i$ be the ball with a orbifold curve of degree p passing through which we are going to remove to take connected sum. Let $M_i'=M_i-B^3_p$ and $S^2_p=\partial M_i'$. $\pi^{-1}(M_i')=\widetilde{M_i}$ is connected submanifold of M and $\pi_1|_{\widetilde{M_i}}\colon\widetilde{M_i}\to M_i$ is an abelian cover. Now π_i restricted to a connected component of $\partial\widetilde{M_i}$ is the p-fold cyclic branched cover of S^2, hence the number of boundary components of $\widetilde{M_i}$ is d/p. So clearly $\widetilde{M_i}$ is homeomorphic to M with d/p balls removed, and hence does not depend on M_i and π_i. If we then look at $S^2_p=\partial(L(n,m_i)-B^3_p)$ then the preimage under the universal abelian cover of $p_i\colon S^3\to L(n,m_i)$ of $L(n,m_i)-B^3_p$, is S^3 with n/p balls removed. Let $\widetilde{M}$ be the manifold constructed the following way. Take n/p copies of $\widetilde{M_i}$ and d/p copies of S^3 with n/p balls removed. Then glue each of the $\widetilde{M_i}$ to each of the S^3's exactly once, to form $\widetilde{M}$. Since π_i and the universal abelian cover map from S^3 to $L(n,m_i)$ agrees on boundary components, we get an abelian cover $\tilde{\pi}_i\colon\widetilde{M}\to L(n,m_i)\#_pM_i$ of degree nd/p, by letting $\tilde{\pi}_i$ be equal to π_i on each of the $\widetilde{M_i}$ components and to the p_i on the S^3 components.

Using Meyer-Vietoris sequence we get that $|H_1^{orb}(L(n,m_i)\#_p M_i)| = nd/p$ hence $\tilde{\pi}_i \colon \widetilde{M} \to L(n,m_i)\#_p M_i$ is the universal abelian cover, which proves the lemma since the homeomorphism type for $\widetilde{M}$ only depends on M, n, d and p. $\square$

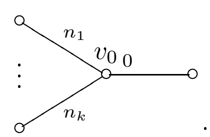
Remark that the above theorem is not true if we took connected sum along spheres with different degrees of the orbifold points, e.g $L(6,3)\#_3 L(6,3)$ has universal abelian cover $S^1 \times S^2$, but the universal abelian cover of $L(6,3)\#L(6,3)$ has first homology group of rank 25. Since the splice diagram cannot see the orbifold curve we are going to take connected sum along, this forces us to make the assumption on our graph orbifolds in the theorem below. Alternatively one could make a new definition of splice diagram, where at leaves of weight zero one specifies the degree of the orbifold curve in the solid torus corresponding to the leaf. The theorem then holds for all graph orbifolds with this invariant.

Theorem 6.2. *If M and M' are two rational homology sphere graph orbifolds having the same splice diagram Γ, and assume that all solid tori corresponding to leaves of weight zero do not have orbifold curves. Then $\widetilde{M}$ is homeomorphic to $\widetilde{M}'$, where $\pi \colon \widetilde{M} \to M$ and $\pi' \colon \widetilde{M}' \to M'$ are the universal abelian orbifold covers.*

Proof. We will prove the theorem by inductively constructing $\widetilde{M}$ only using information from the splice diagram.

For the case with one node, this is mostly the theorem from [Neu83a] and [Neu83b] cited above, since every one-node graph orbifold is a S^1 fibered orbifold, if we have no edge weight of 0. So we have to consider the case of a one-node splice diagram with a edge weight of 0.

Let M be a orbifold with the following splice diagram



This orbifold is a S^3 connect summed along smooth S^2's with the orbifold quotients of S^3 by $\mathbb{Z}/n_i\mathbb{Z}$ $L(n_1,q_1),\dots,L(n_k,q_k)$, where the pair (n_i,q_i) is the Seifert invariant of the i'th singular fiber.

One sees this the following way. The leaf with edge weight zero means that the fibers of the piece corresponding to the central node bounds a meridional disc in the solid torus Z corresponding to the leaf of weight zero. Take two fibers F_1 and F_2 and a simple path p between them in the boundary of the Z. Then the region $B \subset Z$ bounded by all the fibers intersecting p and the meridional discs bounded by F_1 and F_2 is a ball. We can now extend B to the boundary of the solid torus L corresponding to the leaf of weight n_i. So by this, part of the boundary of B is a annulus of fibers in ∂L. Now $L \bigcup B$ is a solid torus glued to a ball along a strip cross a knot which is a representative of a non trivial homology class of the boundary of L, and hence clearly has boundary S^2. Another way to see this is that the boundary is an annulus union 2 discs. $L \bigcup B$ includes a singular fiber, hence it is not a ball, and therefore the S^2 is a separating sphere, and $M = (L \bigcup B)\#(M - L \bigcup B)$.

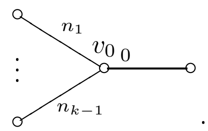
What is left is just to see what $L \bigcup B$ is. To see this we see that the complement of B in Z is a ball. So gluing this ball to $L \bigcup B$ we get the same orbifold as if we glued L to Z, and since Z does not have any singular fibers it is a quotient of S^3 with a orbifold curve with Seifert invariant (n_i,q_i).

So by doing this for each of the leaves with non zero weight, we get that M is connected sum of the S^3 quotients $L(n_1,q_1),\dots,L(n_k,q_k)$ and a central piece M'. What is left is to see that the central piece is S^3. If we glue a ball in $M - L \bigcup B$

to make the closed manifold M', we see that M' is Z glued to a solid torus, and the gluing map is the same as when we glued Z to $M - Z$. Since the weight of the leaf corresponding to Z was zero, it implies that a fiber of $T^2 = \partial(M - Z)$ is a generator of $H_1(T^2)$ and is glued to a meridian of Z and a simple closed curve c corresponding to the other generator is glued to a longitude of Z. This is because that is how one specifies the gluing of Z to a solid torus to get the decoration of the splice diagram, the glued manifold here being $S^1 \times S^2$ since the weight is zero. But gluing two solid tori according to the gluing of M and Z described above creates a S^3.

To show that the universal abelian cover of M is determined by the splice diagram, we do induction in the number of S^3 quotients in M, i.e. the number of leaves of the splice diagram of M. If there is only one S^3 quotient, then S^3 connect sum $L(n, q)$ is just $L(n, q)$, so the universal abelian cover of M is just S^3 and the covering map has degree n, hence determined by the splice diagram.

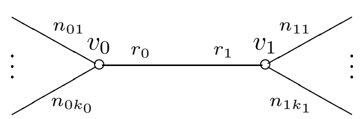
Let M' be the connected sum of S^3 with $L(n_1, q_1), \dots, L(n_{k-1}, q_{k-1})$. Then M' has splice diagram as follows



So by induction the universal abelian cover $\widetilde{M}'$ of M' and the degree for this universal abelian cover d is determined by the splice diagram. We also have that $M = L(n_k, q_k)\#M'$ so by Lemma 6.1 the universal abelian cover of M is determined by $\widetilde{M}'$, n_k and the degree of the cover $\widetilde{M}' \to M'$ (remember in this case $p = 1$). But all this information is given by the splice diagram, since the splice diagram of M' is determined by the splice diagram of M.

This completes the one node case. For more than one node we will reduce our case to one with fewer nodes by cutting along a torus in M corresponding to an edge joining two nodes in Γ. This is more complicated than when we cut along a sphere since what we cut along is now not simply covered once by itself, but may be multiply covered, and the gluing of 2–torus boundary components is not trivial as it is with 2–spheres.

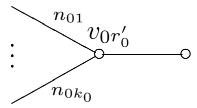
Let us assume that M has splice diagram Γ with $n > 1$ nodes. We look at a edge between two nodes of the form



Let $T^2 \subset M$ be the separating torus, corresponding to the edge we have chosen. Let $M_i^\circ \subset M - T^2$ be the connected component containing the node v_i, and let $M_i = M_i^\circ \cup T^2$. M_0 and M_1 are graph orbifolds with one boundary torus each. $\pi|_{\pi^{-1}(M_i)} \colon \pi^{-1}(M_i) \to M_i$ is a (possibly disconnected) abelian covering. Let $\widetilde{M}_i \subset \pi^{-1}(M_i)$ be a connected component. Then $\pi|_{\widetilde{M}_i} : \widetilde{M}_i \to M_i$ is a connected abelian covering.

To describe the covering $\pi|_{\widetilde{M}_i} : \widetilde{M}_i \to M_i$ we are going to construct a closed graph orbifold M_i', with $M_i \subset M_i'$, such that if $p \colon \widetilde{M}_i' \to M_i'$ is the universal abelian cover, then $p|_{p^{-1}(M_i)} \colon p^{-1}(M_i) \to M_i$ is equal to $\pi|_{\widetilde{M}_i} : \widetilde{M}_i \to M_i$, i.e., $\widetilde{M}_i = p^{-1}(M_i)$ and the maps $p|_{\widetilde{M}_i}$ and $\pi|_{\widetilde{M}_i}$ agree.

We first look at M_0'. We will construct it from M_0 by gluing a solid torus in the boundary of M_0, in a way we will now explain. Now M_0' has splice diagram



where every weight on the left is as in the splice diagram of M if it does not see the edge we are cutting along. We want to determine r_0' and the other weights that see the edge we are cutting along so that the universal abelian cover has the desired properties.

To determine $\widetilde{M}_0$ we use that the components of a non connected abelian cover are determined by the map from H_1^{orb} of the base space to the abelian group which determines the non connected cover. So in our case $\widetilde{M}_0$ is determined by $H_1^{orb}(M_0) \to H_1^{orb}(M)$. One makes M_0' by gluing in a solid torus with a orbifold curve of degree p such that the generator of $\ker\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right)$ is the curve that get killed, i.e. $\ker\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right) = \ker\left(H_1^{orb}(M_0) \to H_1^{orb}(M_0')\right)$, by gluing the primitive element α such that $\langle p\alpha\rangle = \ker\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right)$ to a meridian of the solid torus with a orbifold curve of degree p and a simple closed curve with intersection 1 with the generator to a longitude. This ensures that $\widetilde{M}_0$ embeds into the universal abelian cover of M_0'. We also gets that

$$\begin{aligned} H_1(M_0') &= \operatorname{Im}\left(H_1^{orb}(M_0) \to H_1^{orb}(M_0')\right) = H_1^{orb}(M_0)/\ker\left(H_1^{orb}(M_0) \to H_1^{orb}(M_0')\right) \\ &= H_1^{orb}(M_0)/\ker\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right) = \operatorname{Im}\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right). \end{aligned}$$

This last fact is what we want to use to find the splice diagram of M_0', so we need to determine $\operatorname{Im}\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right)$.

We first determine $\ker(H_1^{orb}(M_0) \to H_1^{orb}(M))$, by looking at the Meyer Vietoris sequence of the covering of M by M_0 and M_1.

$$\cdots \to H_2^{orb}(M) \to H_1(T^2) \to H_1^{orb}(M_0) \oplus H_1^{orb}(M_1) \to H_1(M) \to \dots \tag{34}$$

Since we have Poincare duality with rational coefficients, it follows that $H_2^{orb}(M)$ is finite, so $H_1^2(T^2) = \mathbb{Z}\oplus\mathbb{Z}$ injects into $H_1^{orb}(M_0)\oplus H_1^{orb}(M_1)$. Hence $\ker(H_1^{orb}(M_0) \to H_1^{orb}(M))$ is equal to the intersection of $H_1^{orb}(M_0)$ with $\mathbb{Z}\oplus\mathbb{Z}$, by using again that the rational homology of M_0 is just as the rational homology of manifold, it follows that $H_1^{orb}(M_0)$ is rank one. Therefore $\ker(H_1^{orb}(M_0) \to H_1^{orb}(M)) = \mathbb{Z}$ and is generated by a class in the boundary of M_0.

Let $Q_0 \in H_1(M_0)$ be a representative of the homology class of a section of the fibration of T^2, and let $F_0 \in H_1(M_0)$ be a representative of the class of the fibers of the Seifert fibered piece corresponding to the node v_0 in M_0. Then some homology class T^2 given by $r_0'Q_0 + s_0F_0$ is the class that gets killed when we glue in M_1, so it represents the generator of $\ker(H_1^{orb}(M_0) \to H_1^{orb}(M))$.

We have that $|H_1^{orb}(M_1)/\langle F_0\rangle| = r_0$ by the definition of splice diagram, since $H_1^{orb}(M_1)/\langle F_0\rangle = H_1^{orb}(M_1/F_0)$. Let $|H_1^{orb}(M_1)/\langle F_0, Q_0\rangle| = d_1$. Then, since $H_1^{orb}(M_1)/\langle F_0, Q_0\rangle = H_1^{orb}(M_1/\partial)$, d_1 is equal to the ideal generator defined by Neumann and Wahl in [NW05a], which is an invariant of the splice diagram. This is their theorem 12.9, whose proof also works for graph orbifolds. This implies that the order of Q_0 in $H_1^{orb}(M_1)/\langle F_0\rangle$ is r_0/d_1. Since $r_0'Q_0 + s_0F_0 = 0$ in $H_1^{orb}(M_1)$ we get that $r_0'Q_0 = 0$ in $H_1^{orb}(M_1)/\langle F_0\rangle$, so $r_0' \mid r_0/d_1$.

We also have the following map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}\langle F_0\rangle \arrow[r] & H_1^{orb}(M_1) \arrow[r] & H_1^{orb}(M_1)/\langle F_0\rangle \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z}\langle F_0\rangle \arrow[r] \arrow[u, "\cong"] & (\mathbb{Z}\times\mathbb{Z})/\langle r_0'Q_0 + s_0F_0\rangle \arrow[r] \arrow[u] & \mathbb{Z}/(r_0') \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

since the left map is an isomorphism and middle map is injective, it follows that the right map is injective too, hence $r_0' = r_0/d_1$. Also note that $H_1^{orb}(M_1, \partial) = H_1^{orb}(M, M_0) = H_1^{orb}(M)/\operatorname{Im}\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right)$, so by taking the order of the groups, we get that $d_1 = d/|\operatorname{Im}\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right)|$, and since $H_1^{orb}(M_0') = \operatorname{Im}\left(H_1^{orb}(M_0) \to H_1^{orb}(M)\right)$, one gets that $|H_1^{orb}(M_0')| = d/d_1$.

Now for the other edge weights which see the edge we are cutting along, we start by determining the edge weight on an edge which connects to our chosen node. Since the fiber intersection number corresponding to the edge is the same in M and M_0', we get by 3.2 the following equations

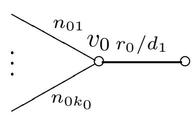
$$\frac{|D|}{|H_1^{orb}(M)|} = \frac{|D'|}{|H_1^{orb}(M_0')|} \tag{35}$$

where D is the edge determinant corresponding to the edge in M and D' the edge determinant corresponding to the edge in M'. Since $|H_1^{orb}(M_0')| = |H_1^{orb}(M)|/d_1$ by the above calculation, we get that $|D'| = |D|/d_1$. Since $r_0' = r_0/d_1$, the definition of edge determinants now gives that the changed edge weight on the edge we are looking at has also been divided by d_1. Continuing inductively we see that all edge weight that sees the edge are divided by d_1.

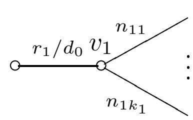
Note that if the curve we kill when we glue in the solid torus in M_0 is a multiple of a primitive element, then we get a orbifold curve in the glued in torus. Now this is not a problem if the weight on the edge is non zero, so let us consider the case when the edge weight $r_0 = 0$. The curve we kill is the curve that bounds in M_1, but $r_0 = 0$ implies that a fiber in ∂M_0 bounds in M_1, so the curve we kill is a fiber. But fibers are primitive elements, so we do not get a orbifold curve in this case. This insures that M_1' satisfies the hypothesis of the theorem.

By a similar argument $r_1' = r_1/d_0$ where $d_0 = |H_1^{orb}(M_0, \partial)| = |H_1(M, M_1)|$.

All this implies that the splice diagram Γ_0 of M_0' look likes



and the splice diagram Γ_1 for M_1' is



Where all edge weights which see the cut edge are gotten from the old weight by dividing by d_1 in the first case and d_0 in the second case.

Since d_0 and d_1 are the ideal generators defined in 12.8 of [NW05a], they are completely determined by Γ, since they are generators of ideals defined by numbers coming from the decorations of Γ. Now the splice diagrams Γ_0 and Γ_1 have at most $n-1$ nodes each, since we get them by removing at least one node of Γ, hence by induction $\widetilde{M_0'}$ is determined by Γ_0 and $\widetilde{M_1'}$ are determined by Γ_1. Since Γ_0 and Γ_1 are determined by Γ, this implies that $\widetilde{M_0'}$ and $\widetilde{M_1'}$ are determined by Γ, and therefore by the construction of M_0' and M_1', we have that $\widetilde{M_0}$ and $\widetilde{M_1}$ is determined by Γ. Next we want to see that the splice diagram determines the number of components of $\pi^{-1}(M_0)$ and $\pi^{-1}(M_1)$ and which components are glued to which. The group of permutations of the components of $\pi^{-1}(M_i)$ is given by $H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(M_i) \to H_1^{orb}(M))$. But $H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(M_i) \to H_1^{orb}(M)) = H_1^{orb}(M, M_i)$, so the number of components of $\pi^{-1}(M_i)$ is the order of $H_1^{orb}(M, M_i)$, which we have seen before is d_i, and only depends on the splice diagram.

To see which components are glued together, we notice that all these gluings are along tori in $\pi^{-1}(T^2)$, so the gluing are specified by the group of permutations of the components of $\pi^{-1}(T^2)$. So we look at $H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(T^2) \to H_1(M))$, which is the same as $H_1^{orb}(M, T^2)$. By excision this is $H_1^{orb}(M/T^2)$. Now $M/T^2 = M_1/T^2 \vee M_2/T^2$, so the group of permutations of $\pi^{-1}(T^2)$, is given by $H_1^{orb}(M_1, T^2) \oplus H_1^{orb}(M_2, T^2)$. Since the group of permutations of the tori is just the product of the permutations of the group of permutations of the components of each sides, it follows that a component on the one side is glued to each component on the other side exactly once.

The last last thing to see is that the gluing of a component of $\pi^{-1}(M_0)$ to a component of $\pi^{-1}(M_1)$ is specified by the splice diagram. Let $\widetilde{M}_0$ and $\widetilde{M}_1$ be components on each of the sides. To specify the gluing, we show that the splice diagram determines two distinct essential curves up to homotopy in each component T^2_{0i} of $\partial\widetilde{M}_0$ and in each components T^2_{1j} of $\partial\widetilde{M}_1$. The curves are the same in each of the T^2_{0i}'s and likewise in each of the T^2_{1i}'s. If the curves in T^2_{0i} are called F_0, C_0 and in T^2_{1j} the curves are F_1, C_1, then when a T^2_{0i} is glued to a T^2_{1j}, F_0 is identified with C_1 and C_0 is identified with F_1.

Now $\widetilde{M}_0$ and $\widetilde{M}_1$ are the total spaces of graph orbifolds with boundary completely determined by Γ, hence we have naturally defined fibers $\widetilde{F}_i$ by the restriction of $\pi^{-1}(F_i)$, where F_i are fibers in the boundary of M_i. So these fibers are going to be one of our curves on each side. The other curve in T^2_{ij} is then the curve the fiber from the other side are going to be identified with. So we need to find this curve.

Let $\widetilde{N}_i \subset \widetilde{M}_i$ be the Seifert fibered piece sitting over the S^1 fibered piece of $N_i \subset M_i$ corresponding to the node v_i. Now $\widetilde{N}_i$ is a piece of the JSJ decomposition of $\widetilde{M}$ and therefore has a rational euler number $\widetilde{e}_i$. This is the rational euler number of the closed Seifert fibered manifold one gets by gluing solid tori in $\partial\widetilde{N}_i$, specified by the gluings of $\widetilde{N}_i$ to the other pieces of the JSJ decomposition as described in beginning of Section 3. We are going compute $\widetilde{e}_i$, and then use this to specify the other curve in each T^2_{ij}. To do this we can assume by induction, that a simple closed curve transverse to the fibration is determined in all the boundary component of $\widetilde{N}_i$ except the boundary components lying over the edge between v_0 and v_1 are specified. But the fibers and the simple closed curves in all the boundary components lying over the edge between v_0 and v_1, will be the same. Since we know $\widetilde{F}_i$ the rational euler number $\widetilde{e}_i$ of $\widetilde{N}_i$ determines a simple closed curve transverse to the fibration in each of the boundary components of $\widetilde{M}_i$.

To compute $\widetilde{e}_0$ we will use the relation between $\widetilde{e}_0$ and the rational euler number of the S^1 fibered piece in the base N_0, i.e., the e_{v_0}. This relation is given by theorem 3.3 of [JN83], and give in our situation that $\widetilde{e}_0 = \frac{b_0}{f_0} e_{v_0}$, where b_0 is the degree of the covering map restricted to the base of the Seifert fibered pieces and f_0 is the degree of π restricted to the fibers. Now $\deg(\pi|_{\widetilde{M}_0}) = b_0 f_0$ and $\deg(\pi|_{\widetilde{M}_0}) = \frac{d}{d_0}$ since d_0 is the number of component of $\pi^{-1}(M_0)$. This implies that $\widetilde{e}_0 = \frac{d}{d_0 f_0^2} e_{v_0}$.

To calculate f_0 notice that $f_0 = |\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))|$, and since

$$|H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))| = |H_1^{orb}(M)|/|\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))|$$

we get that $f_0 = |H_1^{orb}(M)|/|H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))|$, so we want to calculate $|H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))|$.

Now $H_1^{orb}(M)/\operatorname{Im}(H_1^{orb}(F_0) \to H_1^{orb}(M))$ is is the same as $H_1^{orb}(M, F_0) = H_1^{orb}(M/F_0)$, so we need to find $|H_1^{orb}(M/F_0)|$. M/F_0 is M with the fibers collapsed at the piece N_0; this is the same as gluing in disks in the fibers of N_0. This then implies that we glue a disk to a simple closed curve transverse to the fibration of all the S^1–fibered pieces glued to N_0 along a torus. This implies that

$H_1^{orb}(M/F_0) = \big(A_1 \times A_2 \times \cdots A_{k_0} \times G\big)/\ (a_1, a_2, \ldots, a_{k_0}, g)$, where the group A_j is the first orbifold homology group of the graph orbifold one gets by cutting along a torus corresponding to the $0j$'th edge between the N_0 and the j'th piece and gluing in a solid torus. In particular $|A_j| = n_{0j}$. The group G is the first orbifold homology group one gets by cutting along the edge between v_0 and v_1 and gluing in a solid torus in the part not containing v_0. Hence $|G| = r_0$. The $a_j \in A_j$ are the elements that corresponds to the singular fibers over the disk we just glued in; in particular $A_j/\langle a_j\rangle$ is the first orbifold homology group of M with everything except the part across the edge $0j$ collapsed. Thus $|A_j/\langle a_j\rangle|$ is the ideal generator d_{0j} corresponding to the edge $0j$. The same holds for g and G, especially that $|G/\langle g\rangle| = d_1$.

If none of the A_j and G is infinite, i.e., $n_{0j}, r_0 \neq 0$, then $|H_1^{orb}(M/F_0)| = (r_0 \prod_{j=0}^{k_0} n_{0j})/\operatorname{lcm}(n_{01}/d_{01}, \ldots, n_{0k_0}/d_{0k_0}, r_0/d_1)$. Assume that A_l is infinite, then all the other groups are finite, and we have the following exact sequence

$$0 \to G \times \prod_{\substack{j=1 \\ j\neq l}}^{k_0} A_j \to \big(A_1 \times \cdots A_{k_0} \times G\big)/\ (a_1, \ldots, a_{k_0}, g) \to A_l/\langle a_l\rangle \to 0$$

where the $\big(A_1 \times \cdots A_{k_0} \times G\big)/\ (a_1, \ldots, a_k, g) \to A_l/\langle a_l\rangle$ map is projection, and $G \prod_{\substack{j=1 \\ j\neq l}}^{k_0} A_j$ is the kernel of this map. This implies that in this case $|H_1^{orb}(M/F_0)| = d_{0l} r_0 \prod_{\substack{j=1 \\ j\neq l}}^{k_0} n_{0j}$. If G is infinite we in the same way get that that $|H_1^{orb}(M/F_0)| = d_1 \prod_{j=1}^{k_0} n_{0j}$. In all cases we see that $|H_1^{orb}(M/F_0)| = \lambda_0$ only depends on the splice diagram, since the ideal generators only depends on the splice diagram.

So now we get that $f_0 = |H_1^{orb}(M)|/|H_1^{orb}(M/F_0)| = \frac{d}{\lambda_0}$, hence $\widetilde{e}_0 = \frac{\lambda_0}{d_0 d} e_{v_0}$. But the value of e_{v_0} is given by proposition 5.6, and the formula given there shows that e_{v_0} is d times a number given by the splice diagram, hence $\frac{\lambda_0}{d_0 d} e_{v_0}$ is $\frac{\lambda_0}{d_0}$ times a number only depending on Γ, hence $\widetilde{e}_0$ only depends on Γ. We can in the same way calculate $\widetilde{e}_1$ and see that it also only depends on Γ. This implies that the splice diagram specifies a simple closed curve C_i transverse to the fibration of the T_{ij}^2's, which in particular is not null homologous.

Now the gluing of $\widetilde{M}_0$ to $\widetilde{M}_1$ is specified by identifying $\widetilde{F}_0$ with C_1 and $\widetilde{F}_1$ with C_0. But since the $\widetilde{F}_i$'s and the C_i's are determined by Γ, the gluing is determined by Γ, and hence $\widetilde{M}$ is determined by Γ. □

Corollary 6.3. *Let M be a rational homology sphere graph manifold with splice diagram $\Gamma(M)$, such that around any node in $\Gamma(M)$ the edge weights are pairwise coprime, then the universal abelian cover of M is an integer homology sphere.*

Proof. It is shown in [EN85] that a splice diagram with pairwise coprime edge weights at nodes is the splice diagram for an integer homology sphere. So given a splice diagram satisfying the assumption there is a integer homology sphere M' with splice diagram $\Gamma(M)$, hence M' is the universal abelian cover of M by 6.2. □

This actually gives a way to construct the universal abelian cover for any rational homology sphere graph manifold that has a splice diagram with pairwise coprime edge weights at nodes, since [EN85] describes how to construct the integral homology sphere by splicing, and to construct a plumbing diagram for it in the case where we have positive decorations at nodes. If we want a construction of the plumbing diagram in the case with negative decorations at nodes it follows from theorem 3.1 in [Neu89a] which give us that given a splice diagram $\Gamma(M)$ as above, one can construct an unimodular tree $\Delta(M)$, which will be the plumbing diagram.

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